§5.2 Volumes

Key points:

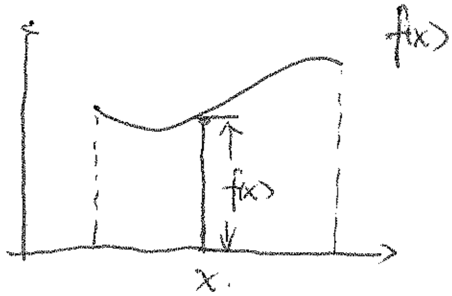
① $V = \pi \int_a^b \text{Radius}^2 \, dx$ Disk

② $V = \pi \int_a^b \text{Outer Radius}^2 - \text{Inner Radius}^2 \, dx$. Washer

(① and ②: Solid of Revolution)

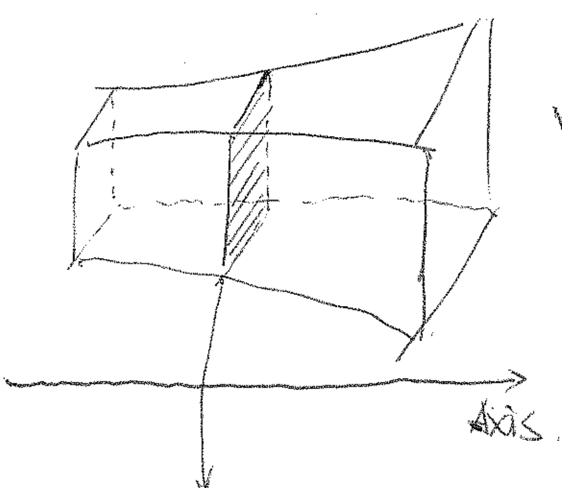
③ $V = \int_a^b \text{Side length}^2 \, dx$ - Pyramid with Square Cross-Section.

★★ ④ $V = \int_a^b A(x) \, dx$. General Cross Section with Area $A(x)$



$$\text{Area} = \int_a^b \underbrace{f(x)}_{\text{length(Height) of "Cross Section"}} dx = \int_a^b \text{length} \, dx.$$

⇓ Generalization

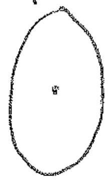


Cross Section (Perpendicular to axis)

$$\text{Volume} = \int_a^b \underbrace{A(x)}_{\text{Area of the Cross-Section}} dx$$

Two Types of Cross-Sections

- Disk.



Rotating a plane curve/region.

(Washer = two disks

- Square Cross-Section

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e.g.1 (Disk)

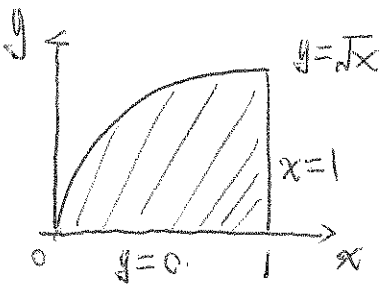
① Sketch the region bounded by $y=\sqrt{x}$, $y=0$, $x=1$.

② Rotate the region w.r.t. x-axis, describe the solid.

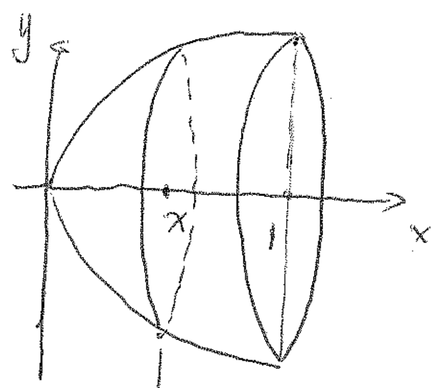
③ Set up an integral for the volume of the rotating solid.

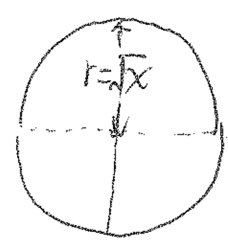
④ Find the volume.

①



②



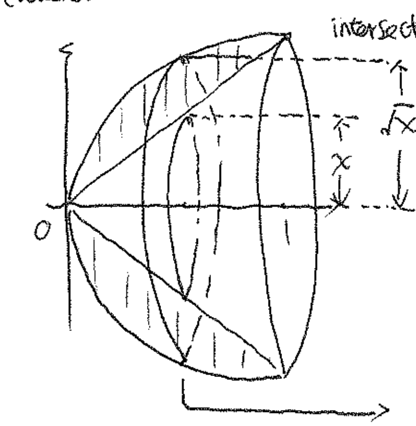


Cross Section is a DISK with radius $r=\sqrt{x}$. $\text{Area} = \pi \cdot r^2$
$= \pi (\sqrt{x})^2$

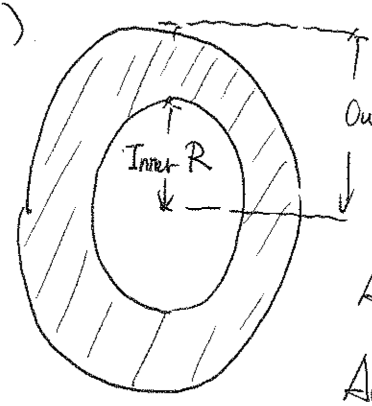
③ $$\boxed{V = \int_0^1 \pi \cdot (\sqrt{x})^2 \, dx.}$$

④ $$V = \int_0^1 \pi \cdot x \, dx = \pi \cdot \frac{1}{2} x^2 \Big|_0^1 = \pi \cdot \frac{1}{2} \cdot 1^2 - \pi \cdot \frac{1}{2} \cdot 0^2$$
$$= \boxed{\frac{1}{2} \cdot \pi}$$

e.g.2 (Washer) Region enclosed by $y=\sqrt{x}$, $y=x$, rotated w.r.t. x-axis.



Project the cross-section on the plane



Outer Radius $R_{outer}=\sqrt{x}$.

$R_{inner}=x$

Area of Cross-Section:

$$A(x)=\pi\cdot R_{outer}^2-\pi\cdot R_{inner}^2$$
$$=\pi\cdot(\sqrt{x})^2-\pi\cdot x^2$$

$$\text{Volume}=\int_0^1 A(x)\cdot dx=\int_0^1[\pi\cdot\sqrt{x}^2-\pi\cdot x^2]\cdot dx.$$ (Set-up integral only).

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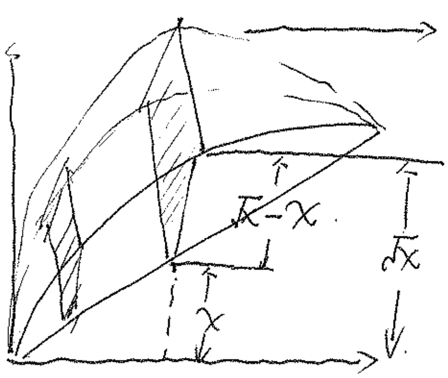
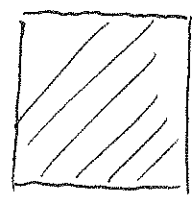
e.g.3. (Pyramid) Let the region be the same as in eg.2. The cross section is a square perpendicular to x-axis with side in X-Y plane.

① Describle the solid. ② Find (Set up) the volume.



Cross-Section is a Square with side $L=\sqrt{x}-x$.



$L=\sqrt{x}-x$.

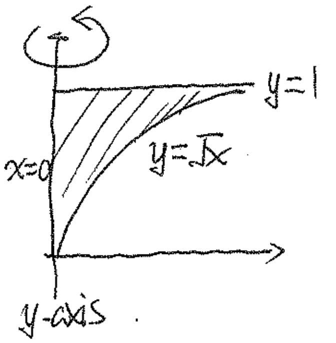
Area of C.S: $A(x)=L^2=(\sqrt{x}-x)^2$

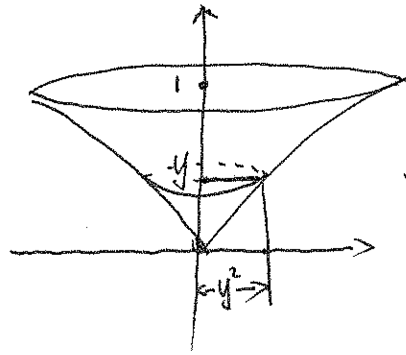
$$\text{Volume}=\int_0^1 A(x)\cdot dx=\boxed{\int_0^1(\sqrt{x}-x)^2\cdot dx}$$

☆ Rotating about y-axis.

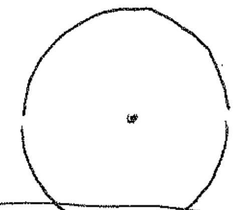
e.g. 4. Region enclosed by $y=\sqrt{x}$, $x=0$, $y=1$

Rotate the region about y-axis. Find the volume of the solid.





Cross. Section



$$\boxed{\text{Radius} = x = y^2}$$

Note: The cross-section is moving along y-direction. The integral is set up with respect to $\boxed{y\text{ - variable}}$.

$$A(y) = \pi \cdot \text{Radius}^2 = \pi \cdot (y^2)^2 = \pi y^4$$

$$V = \int_0^1 A(y) \cdot dy$$

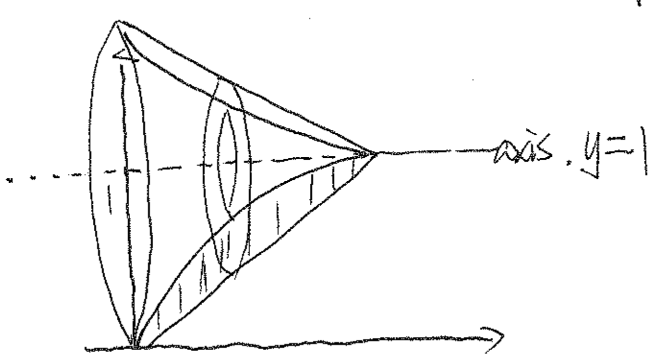
$$= \boxed{\int_0^1 \pi \cdot y^4 \cdot dy}.$$

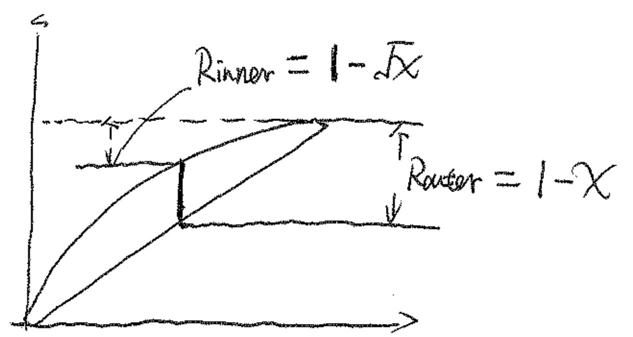
☆☆. Rotating about OTHER axis.

eg. 5. Region: $y=\sqrt{x}$, $y=x$, rotated about $y=1$.

Note: $y=1$ is a HORIZONTAL axis (parallel to x axis)

The integral is set up with r.t. $\boxed{x}$.





Area of Cross-Section: $A(x) = \pi \cdot R_{outer}^2 - \pi \cdot R_{inner}^2$

$$= \pi \cdot (1-x)^2 - \pi \cdot (1-\sqrt{x})^2$$

$$\text{Volume} = \int_0^1 A(x) \cdot dx = \boxed{\int_0^1 \pi \cdot (1-x)^2 - \pi \cdot (1-\sqrt{x})^2 \cdot dx}.$$